Novel Asymptotic Soliton Waves for the Nonlinear Schrödinger Equation with Varying Gain/Loss and Frequency Chirping

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This paper analysis spatial asymptotic waves propagation in nonuniform optical fiber. It finds an appropriate transformation such that the nonlinear variable-coefficient Schrödinger equation transform into the nonlinear Schrödinger equation with varying gain/loss and frequency chirping. It obtains solitonlike and periodic self-similar asymptotic waves by using the transformation. We analyze the evolution properties of some novel self-similar solutions. In addition, the nature of our self-similar asymptotic wave hints to the possibility of designing optical amplifier and focusing of spatial waves to overcome inevitable energy losses while performing in the optical nonlinear media.

Key words: Self-Similar Asymptotic Wave; Nonlinear Schrödinger Equation; Varying Gain/Loss; Frequency Chirping.
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1. Introduction

It is one of the focuses of the optical fibers in the world in recent years to produce and transmit the chirp pulse with high power. A self-similar pulse [1, 2], generated in a dispersion decreasing optical fiber or fiber amplifier with normal group-velocity dispersion, has become a topic of growing interest owing to its attractive characteristics, such as resistance to optical wave breaking, self-similarity in shape, and enhanced chirp linearity. Moreover, its linear chirp facilitates efficient temporal compression. These attractive features lead the self-similar pulse to a wide-range of practical significance. The self-similar pulse has obtained the extensive concern of the foreign counterpart in recent years and has important application prospect in many fields of physics, such as fiber optic communication, nonlinear optics, ultrafast optics and transient optics, etc. Up till now, optical researchers have carried on several theoretical analyses, numerical simulations and experiments to the self-similar pulse and have made a lot of valuable achievements. Among them, the experiments main research the distributing longitudinal of gain parameter of optical fiber, stimulated Raman effect, the properties of the self-similar pulse produced by dispersion decreasing fiber and Bragg grating etc. From the analytical point of view, with the aid of the numerical simulation, they investigate the propagation properties including amplitude, phase, chirp factor and pulse width of optical self-similar pulses for the nonlinear Schrödinger equation with normal group-velocity dispersion. Need to prove, the contents of that study are all carried on under ideal conditions.

In terms of physics, we have supposed that the optical fiber is uniform and the systematic parameters at this moment, such as group-velocity dispersion, nonlinear Kerr effect, third-order dispersion, self-steepening and self-frequency shift are all constants in the whole optical fiber. From the mathematics, the nonlinear Schrödinger equation and high-order nonlinear Schrödinger equation that we study are all ordinary differential equations. In fact, under such ideal conditions, the transmission properties of the optical soliton in the single-mode optical fiber have already been widely studied. Authors have studied different types of Schrödinger equations and discuss the transmission properties of the picosecond and femtosecond pulses in the uniform optical fiber using different methods [3, 4], such as inverse scattering method [5, 6], Hirota method [7], painleve property [8], Darboux transformation [9, 10], ansatz method [11], etc. However, in practical application, the core of the optical fiber is not even. The distance among two adjoined atoms is not constant in the whole optic fiber because of the changing of lattice parameter of the fiber. And the geometric form of the optic fiber changes because of fluctuations of the core diameter of the fiber. The nonuniformity of the optical fiber cause various effects, the optic fiber gain/loss, group velocity dispersion and phase modulation are not all constants [12 – 15]. At this moment, the corresponding transmission model is a nonlinear variable-coefficient Schrödinger equation:

$$\mathrm{I}\frac{\partial \Phi}{\partial z}+\frac{\eta_1(z)}{2}\frac{\partial^2 \Phi}{\partial t^2}+\eta_2(z)|\Phi|^2\Phi=-\mathrm{I}\eta_3(z)\Phi \quad (1)$$

where t is the delay time, $\eta_i(z)$, $i=1\ldots3$ are group-velocity dispersion, non linear Kerr effect, and optical gain/loss, respectively, they are all functions of the normalized distance z.

In recent years, the investigation for the transmission properties of the nonlinear waves in such a nonuniform medium causes great interests of people gradually [16, 17]. In the literature, the nonlinear compression of the chirped soliton without phase modulation has already been discussed in detail. And, under the integrability condition, the nonlinear Schrödinger equation with constant gain/loss and frequency chirping has been reported, too. Another concept is the control and management of soliton. And among them what deserves to be mentioned is the dispersion management soliton. Dispersion management soliton will become the scheme that fiber optic communication system of future generation adopt most probably because of their superior performance [18]. Generally speaking, the dispersion management is the technique used in fiber optic system. It is designed to cope with the periodic dispersion introduced by the optical fiber and offset the loss of the optical fiber, and then the light pulse can be transmitted undistorted in the optical fiber. In terms of mathematics, dispersion management system can be described by a nonlinear variable-coefficient Schrödinger equation or higher-order nonlinear variable-coefficient Schrödinger equation. Therefore, it has certain difficulty to deal with this problem in term of analyzing, a lot of research work is completed through numerical calculation [19]. In recent years, average dispersion management soliton is proposed [20]. In the dispersion management system without loss, a average dispersion management soliton system can be obtained by offsetting the fast chirp. This system can be described by the nonlinear Schrödinger equation with gain/loss and frequency chirping. There are already several reports in this respect recently [21, 22].

In the cases described above, the systems can be described by the nonlinear Schrödinger equation with gain/loss and frequency chirping. Therefore, it is very meaningful to study this equation, and to seek new explains in physical fields [23 – 27]. In addition, it is also an important topic to study the nonlinear variable-coefficient Schrödinger equation. There are also reports in this respect in recent years. In a word, it is very meaningful how to find new solutions and new physical applications of these equation.

Equation (1) has a solution with chirped square phase because of the nonuniformity of the optical fiber. To provide an answer to this, let us scale (1) in the forms:

$$\begin{aligned}\Phi&=p(z)q\sqrt{\frac{\eta_1\alpha_2}{2\alpha_1\eta_2}}\exp\left(\mathrm{I}\frac{M(z)t^2}{2}\right),\\ T&=p(z)t=t\exp\left(-\int_0^z\eta_1(\zeta)M(\zeta)\mathrm{d}\zeta\right),\quad (2)\\ Z&=\int_0^z\frac{\eta_1(\tau)p^2(\tau)}{2\alpha_1}\mathrm{d}\tau,\end{aligned}$$

where $M(z)$ is the chirp parameter, so that (1) becomes the nonlinear Schrödinger equation with varying gain/loss and frequency chirping:

$$\mathrm{I}\frac{\partial q}{\partial Z}+\alpha_1\frac{\partial^2 q}{\partial T^2}+\alpha_2|q|^2q-\beta_1(Z)T^2q+\mathrm{I}\beta_2(Z)q=0,\quad (3)$$

α_1 and α_2 are arbitrary constants, and

$$\beta_1(Z)=\frac{(M'_z+\eta_1M^2)\alpha_1}{p^4\eta_1},$$

and

$$\begin{aligned}\beta_2(Z)=\alpha_1(&\eta_2\eta'_{1z}p+2\eta_2\eta_1p'_z-\eta_1p\eta'_{2z}\\ &+\eta_1^2p\eta_2M+2\eta_1p\eta_2\eta_3)(p^3\eta_1^2\eta_2)^{-1}\end{aligned}$$

are the quadratic phase chirp coefficient and the gain/loss coefficient, respectively. This equation also describes the average dispersion management systems. When the last two terms are omitted this propagation equation reduces to the normal form of nonlinear Schrödinger equation (NLSE), which is integrable (meaning it not only admits N-solitary wave solutions, but the evolution of any initial condition is known in principle). We call these N-solitary wave solutions N-solitons, and mean by this that the solitary waves scatter elastically and asymptotically preserve their shape upon undergoing collisions, just like true solitons. In this paper, we reduce (1) to (3) and derive self-similar asymptotic waves. However, for nonuniform pluses the last two terms are nonnegligible and should be retained. In general, the presence or absence of wave solutions depends on the coefficients appearing in (3), and therefore, on the specific nonlinear and dispersive features of the medium. These self-similar asymptotic waves are usually of great physical importance, because they may hint to the possibility of designing optical amplifier and focusing of spatial waves to overcome inevitable energy losses while performing in the optical network.

2. Self-Similar Asymptotic Waves

In order to find some interesting solutions of (3), we use the following ansatz:

$$\begin{aligned} q(Z,T) &= A(Z,T)\exp \mathrm{I}\phi(Z,T), \\ \phi(Z,T) &= B(Z) + \Theta(Z)T + \varphi(Z)T^2. \end{aligned} \tag{4}$$

Substituting (4) into (3), removing the exponential term, and separating the real and imaginary parts, we obtain

$$\begin{aligned} &A_Z + 2\alpha_1 A_T\phi_T + \beta_2 A = -\alpha_1 A\phi_{TT}, \\ &-A\phi_Z + \alpha_1 A_{TT} - \beta_1 T^2 A + \alpha_2 A^3 = \alpha_1 A\phi_T^2. \end{aligned} \tag{5}$$

Here, we are only interested in solutions that give meaningful depictions of the variants $\beta_1(Z)$ and $\beta_2(Z)$:

$$\beta_2(Z) = G(Z), \quad \beta_1(Z) = -\frac{G^2(Z)}{\alpha_1} - \frac{G'_Z(Z)}{2\alpha_1}, \tag{6}$$

where $G(Z)$ is an arbitrary function. The symmetry group analysis of (5) indicates that a self-similar wave solution to this equation ought to be sought in the form:

$$\begin{aligned} A(Z,T) &= f(Z)W(Z,T) = f(Z)W(\omega) \\ &= f(Z)W[f^2(Z)g(Z)(T-h(Z))], \end{aligned} \tag{7}$$

where

$$g(Z) = \exp\left(\int 2G(Z)\mathrm{d}Z\right). \tag{8}$$

After some lengthy but straightforward algebra, we have a set of first-order differential equations for the width $f(Z)$, the coefficient $\Theta(Z)$, and the beam center $h(Z)$. This set of equations is self-consistent only if the chirp parameter $\varphi(Z)$ obey the constraint about the gain/loss coefficient $\beta_2(Z)$:

$$\varphi(z) = \frac{\beta_2(Z)}{2\alpha_1} = \frac{G(Z)}{2\alpha_1}. \tag{9}$$

The chirp function $\omega_c(T) = -\frac{\partial[\varphi(Z)T^2]}{\partial T} = -\frac{G(Z)}{\alpha_1}T$. This shows, there is a linear relation between chirp function ω_c of the self-similar pulse and the time T. It relates to the dispersion and the gain/loss parameters but have nothing to do with properties of incident pulse. The set of first-order differential equations can be readily solved to obtain the following expressions:

$$\begin{aligned} \Theta(Z) &= \Theta_0 g^{-1}(Z), \\ f(Z) &= f_0 g^{-1}(Z), \\ h(Z) &= g(Z)\int[2\alpha_1\Theta_0 g^{-2}(Z)]\mathrm{d}Z + h_0 g(Z), \end{aligned} \tag{10}$$

where Θ_0, f_0 and h_0 are initial values of the corresponding parameters and $g(z)$ is given by (8). Furthermore, the self-similar wave profile $W(\omega)$ and the phase factor $B(Z)$ are found to satisfy

$$B'_Z(Z) = \alpha_1[-\Theta^2(Z) + g^2(Z)\lambda f^4(Z)], \tag{11}$$

$$W''_{\omega\omega} = \lambda W - \frac{\alpha_2}{f_0^2\alpha_1}W^3, \tag{12}$$

which coincides with the evolution of an enharmonic oscillator with potential

$$U(W) = -\frac{\lambda}{2}W^2 + \frac{\alpha_2}{4f_0^2\alpha_1}W^4.$$

Now we proceed with the coupled amplitude-phase formulation. Equation (12) thus becomes

$$W''_{\omega\omega} = \frac{\mathrm{d}}{\mathrm{d}W}\left[\frac{\lambda}{2}W^2 - \frac{\alpha_2}{4f_0^2\alpha_1}W^4 + P\right].$$

Since

$$W''_{\omega\omega} = \frac{\mathrm{d}}{\mathrm{d}W}\left[\frac{1}{2}(W_\omega)^2\right],$$

we can then write

$$\mathrm{d}\omega = \left[\lambda W^2 - \frac{\alpha_2}{2f_0^2\alpha_1}W^4 + 2P\right]^{-1/2}\mathrm{d}W, \tag{13}$$

where P is an arbitrary constant of integration, which coincides with the energy of the enharmonic oscillator.

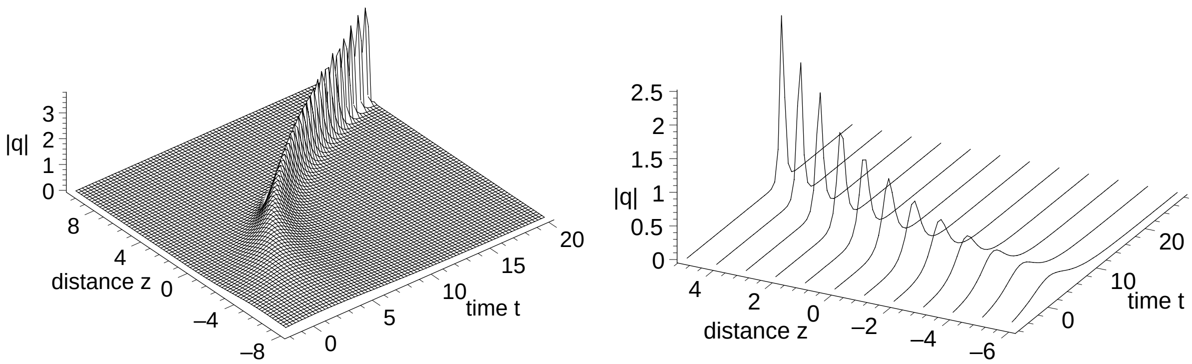

Fig. 1. Evolution of a bright self-similar wave solution (14) in optical fiber with varying gain/loss and frequency chirping for $\alpha_1 = \lambda = f_0 = \Theta_0 = 1$, $\alpha_2 = 2$, $G = -0.05$, $h_0 = 0$.

Integrating (13) for different values of P, we get the amplitude function $W(\omega)$. It is very interesting to look carefully at the above equation. It can be used to construct many types of travelling wave solutions, which include solitary wave solutions, trigonometric function solutions, Jacobian elliptic function solutions, and rational solutions. It is very tedious to write all possible solutions of (13). To avoid more complicated discussion, we only restrict ourselves to several simple and interesting cases.

Case 1. Taking $P = 0$, so from (13) follows that $\mathrm{d}\omega = \left[\lambda W^2 - \frac{\alpha_2}{2f_0^2\alpha_1}W^4\right]^{-1/2}\mathrm{d}W$. We will discuss two subclasses:

(i) For $\lambda > 0$, from (4) and (13) we have

$$q_1(Z,T) = \pm\sqrt{\frac{2\alpha_1\lambda}{\alpha_2}} f_0 f(Z)\operatorname{sech}\left[\pm\sqrt{\lambda} f^2(Z)g(Z)(T - h(Z))\right]\exp[\mathrm{I}B(Z) + \mathrm{I}\Theta(Z)T + \mathrm{I}\varphi(Z)T^2], \quad (14)$$

$$q_2(Z,T) = \pm\sqrt{-\frac{2\alpha_1\lambda}{\alpha_2}} f_0 f(Z)\operatorname{csch}\left[\pm\sqrt{\lambda} f^2(Z)g(Z)(T - h(Z))\right]\exp[\mathrm{I}B(Z) + \mathrm{I}\Theta(Z)T + \mathrm{I}\varphi(Z)T^2], \quad (15)$$

where λ is an arbitrary nonzero constant. Equation (14) is the well-known bright optical soliton solution, while (15) is the soliton profile solution. Here the width and the position of the center of any bright self-similar asymptotic wave are specified by (14) and phase function is given by (10) and (11).

(ii) For $\lambda < 0$, from (4) and (13) we have two singular triangular periodic solutions:

$$q_3(Z,T) = \pm\sqrt{\frac{2\alpha_1\lambda}{\alpha_2}} f_0 f(Z)\sec\left[\pm\sqrt{-\lambda} f^2(Z)g(Z)(T - h(Z))\right]\exp[\mathrm{I}B(Z) + \mathrm{I}\Theta(Z)T + \mathrm{I}\varphi(Z)T^2], \quad (16)$$

$$q_4(Z,T) = \pm\sqrt{\frac{2\alpha_1\lambda}{\alpha_2}} f_0 f(Z)\csc\left[\pm\sqrt{-\lambda} f^2(Z)g(Z)(T - h(Z))\right]\exp[\mathrm{I}B(Z) + \mathrm{I}\Theta(Z)T + \mathrm{I}\varphi(Z)T^2], \quad (17)$$

where λ is an arbitrary nonzero constant.

Case 2. Taking $P \neq 0$, in this case there exist several possibilities depending on the values of the constant P. For example, if $P = f_0^2\lambda^2\alpha_1/4\alpha_2$, then the solution for $q(Z,T)$ reads

$$q_5(Z,T) = \pm\sqrt{\frac{\alpha_1\lambda}{\alpha_2}} f_0 f(Z)\tanh\left[\pm\sqrt{-\frac{\lambda}{2}} f^2(Z)g(Z)(T - h(Z))\right]\exp[\mathrm{I}B(Z) + \mathrm{I}\Theta(Z)T + \mathrm{I}\varphi(Z)T^2], \quad (18)$$

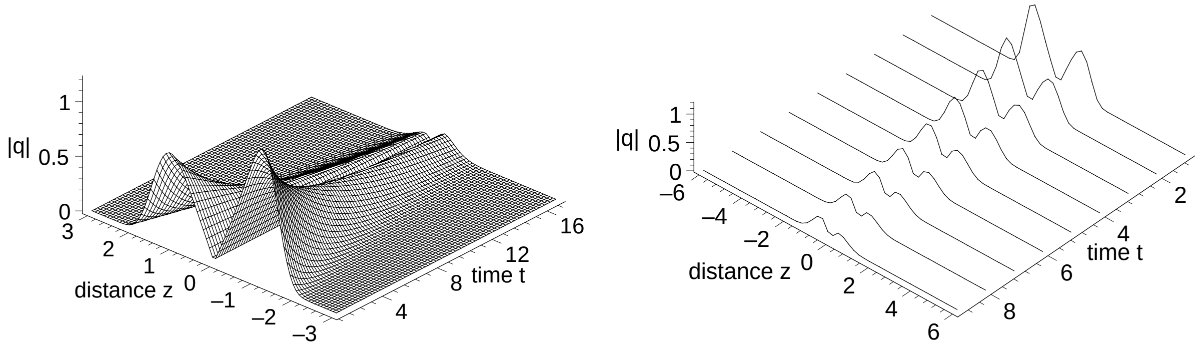

Fig. 2. Evolution of two-solitonlike self-similar wave solution (14) in optical fiber with varying gain/loss and frequency chirping for $\alpha_1 = -0.25$, $\alpha_2 = h_0 = -1$, $\lambda = 2$, $G(z) = -0.5\coth(z)$, $\Theta_0 = f_0 = 5$.

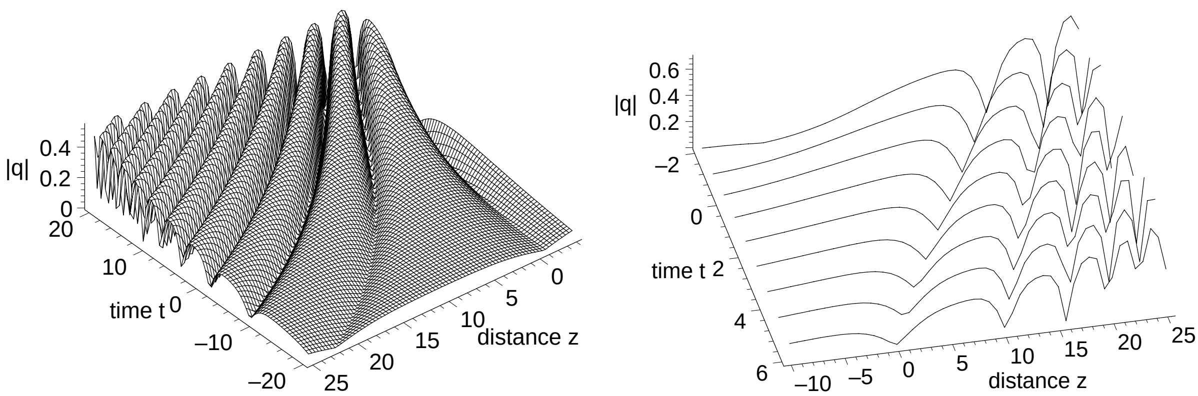

Fig. 3. Evolution of a periodic self-similar wave solution (20) in optical fiber with varying gain/loss and frequency chirping for $\alpha_1 = G = -0.025$, $\alpha_2 = -0.2$, $\lambda = 4$, $\Theta_0 = f_0 = 5$, $h_0 = 0$.

which represents the dark optical soliton for (3), and

$$q_6(Z,T) = \pm\sqrt{-\frac{\alpha_1\lambda}{\alpha_2}}f_0 f(Z)\tan\left[\pm\sqrt{\frac{\lambda}{2}}f^2(Z)g(Z)(T-h(Z))\right]\exp[\mathrm{I}B(Z)+\mathrm{I}\Theta(Z)T+\mathrm{I}\varphi(Z)T^2], \quad (19)$$

whereas if $P = \lambda^2\alpha_1 f_0^2 m^2/(m^2+1)^2\alpha_2$, $q(Z,T)$ can be expressed as the following Jacobian elliptic function:

$$q_7(Z,T) = \pm\sqrt{\frac{2\alpha_1\lambda}{\alpha_2(m^2+1)}}m f_0 f(Z)\mathrm{sn}\left[\pm\sqrt{-\frac{\lambda}{m^2+1}}f^2(Z)g(Z)(T-h(Z))\right]\exp[\mathrm{I}B(Z)+\mathrm{I}\Theta(Z)T+\mathrm{I}\varphi(Z)T^2]. \quad (20)$$

Finally, if $P = \lambda^2\alpha_1 f_0^2(m^2-1)^2/4\alpha_2(m^2+1)^2$, the solution for $q(Z,T)$ is the Jacobian elliptic function:

$$q_8(Z,T) = \pm\sqrt{\frac{\lambda\alpha_1(m^2-1)}{\alpha_2(m^2+1)^2}}f_0 f(Z)\frac{\mathrm{cn}\left[\pm\sqrt{\frac{2\lambda}{m^2+1}}f^2(Z)g(Z)(T-h(Z))\right]}{1+\mathrm{sn}\left[\pm\sqrt{\frac{2\lambda}{m^2+1}}f^2(Z)g(Z)(T-h(Z))\right]}\exp[\mathrm{I}B(Z)+\mathrm{I}\Theta(Z)T+\mathrm{I}\varphi(Z)T^2]. \quad (21)$$

where λ is an arbitrary nonzero constant and m is a modulus. Since $\mathrm{cn}\xi \to \mathrm{sech}\xi, \mathrm{sn}\xi \to \tanh\xi$ as $m \Rightarrow 1$, we see that the Jacobian periodic solutions (20) degenerates to the soliton solutions (18). When $G(Z) = C$ is an arbitrary

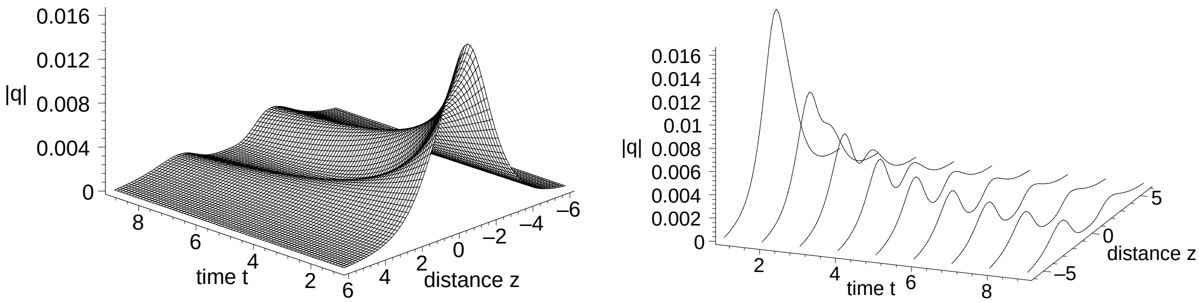

Fig. 4. Fission of the soliton solution (14) in optical fiber with varying gain/loss and frequency chirping for $\alpha_1 = -0.25$, $\Theta_0 = -\alpha_2 = 1$, $\lambda = f_0 = 2$, $G(z) = -0.5\tanh(-z)$, $h_0 = -5$.

constant, we have

$$q(Z,T) = f_0 \exp(-2CZ) W \left[f_0^2 (T \exp(-2CZ) + \frac{\alpha_1 \Theta_0}{2C} \exp(-4CZ) - h_0) \right] \exp \mathrm{I}\phi(Z,T),$$
$$\phi(Z,T) = \frac{C}{2\alpha_1} T^2 + \Theta_0 \exp(-2CZ) T + \frac{\exp(-4CZ)\alpha_1(\Theta_0^2 - \lambda f_0^4)}{4C}. \quad (22)$$

What merits attention is that the nonlinear wave equation in [28] is a special case of (3) with $\alpha_2 = 1$, $\beta_1 = \beta_2 = -\alpha_1 = -1/2$. In the condition $G = -0.05$, from our results in this paper q_1, q_5, we can get the similar results as in the literature [28]. Figure 1 shows the evolution of a bright self-similar wave. The amplitude function is

$$A(Z,T) = \exp\left(\frac{Z}{5}\right) \operatorname{sech}\left[\exp\left(\frac{Z}{5}\right) T - 5 \exp\left(\frac{2Z}{5}\right)\right], \quad (23)$$

and the phase $\phi(Z,T)$ is

$$\phi(Z,T) = -0.025T^2 + \exp\left(\frac{Z}{10}\right) T.$$

The nature of these self-similar asymptotic waves hints to the possibility of designing an amplifier. The proposed device is expected to operate as follows. A linear phase chirp is imprinted on a fundamental spatial wave using an appropriate phase mask placed at the entrance to nonuniform optical fiber. If the amplifier gain satisfies the condition given in (6), the entering phase-chirped spatial wave propagates inside the amplifier as a self-similar asymptotic wave found in this paper and is thus compressed as it is amplified, while preserving its shape. At the exit of the amplifier, a second phase mask is used to remove the phase chirp. The resulting beam is an amplified. However, the constants α_1, α_2 and the functions $\beta_1(z)$, $\beta_2(z)$ are arbitrary in (3) in this paper. Just the alternative of $G(z)$ makes the solutions for (3) more abundant. If we choose $G(z)$ properly, we can obtain several novel excitations of (3). Figures 2 – 4 display three typical self-similar asymptotic waves under the different initial conditions. Figure 2 shows that the two-solitonlike self-similar wave propagates without changing its form and the peak value of energy reduces or increases in the form of index. Generally speaking, a self-similar wave has an ability to resist splitting. However, what is interesting is, in the following special example of $G(z)$ in Figure 4, one soliton has fissed slowly in the course of advancing.

From solutions (14) – (21) we know that the presence of nonlinearity of (3) is essential for these self-similar asymptotic waves to exist. Indeed, the absence of the nonlinearity makes no bound solutions existence to (3). For example, when $\alpha_2 = 0$, we have the solution:

$$q_9(Z,T) = f(Z)[e_0 + e_1 \cosh(f^2(Z) g(Z)(T - h(Z)))] \cdot \exp[\mathrm{I}B(Z) + \mathrm{I}\Theta(Z)T + \mathrm{I}\varphi(Z)T^2], \quad (24)$$

and $f(Z)$, $h(Z)$, $B(Z)$, $\Theta(Z)$, $\varphi(Z)$ are functions given in (10) and (11), and Θ_0, f_0 obey the following constraint:

$$\Theta_0 = \frac{1}{2}\sqrt{2f_0^2 - 2\sqrt{f_0^4 - 4}}, \quad (25)$$

where e_0, e_1, f_0 are arbitrary constants.

3. Summary

In summary, we have studied the Schrödinger equation described for transmission of the pulse in the nonuniform optical fiber or an average dispersion management system. We show analytically that spatial self-similar waves can propagate in the optical fiber with varying gain/loss and frequency chirping. The intensity profiles of the novel waves are identical. Our studies reveal that the pulse expands or compresses when it is being transmitted in the nonuniform optical fiber because the existence of the frequency chirping coefficient. At the same time, the peak value of energy reduces or increases in the form of index, different from the transmission property of the pulse in the uniform optical fiber. In particular, our results shed light on the interesting connection between self-similar waves and solitons existing in nonuniform nonlinear media, the discovered self-similar waves can be used in a promising scheme for the amplification and focusing of spatial solitons in future optical network.

[1] G. I. Barenblatt, Scaling, Self-Similarity, and Intermediate Asymptotics, Cambridge University Press, Cambridge 1996.

[2] S. M. Richardson, Fluid Mechanics, Hemisphere Publishing Corporation, New York 1989.

[3] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP. **34**, 62 (1972).

[4] D. Mihalache, L. Torner, F. Moldoveanu, N. C. Panoiu, and N. Truta, Phys. Rev. E **48**, 4699 (1993).

[5] R. Hirota, J. Math. Phys. **14**, 805 (1973).

[6] M. Gedalin, T. C. Scott, and Y. B. Band, Phys. Rev. Lett. **78**, 448 (1997).

[7] D. Mihalache, N. Truta, and L. C. Crasovan, Phys. Rev. E **56**, 1064 (1997).

[8] V. B. Matveev and M. A. Salle, Darboux Transformation and Solitons, Springer-Verlag, Berlin 1991.

[9] N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, Theor. Math. Phys. **72**, 809 (1987).

[10] V. I. Karpman, J. J. Rasmussen, and A. G. Shagalov, Phys. Rev. E **64**, 026614 (2001).

[11] F. Abdullaeev, Theory of solitons in inhomogeneous media, Wiley, New York 1994.

[12] K. J. Nakkeeran, Phys. Rev. E **62**, 1313 (2000).

[13] R. Ganapathy, V. C. Kuriakose, and K. Porsezian, Opt Commun. **194**, 299 (2001).

[14] M. N. Vinoj, V. C. Kuriakose, and K. Porsezian, Chaos, Solitons, and Fractals. **12**, 2569 (2001).

[15] R. Ganapathy and V. C. Kuriakose, Chaos, Solitons, and Fractals **15**, 99 (2003).

[16] J. D. Moores, Opt. Lett. **21**, 555 (1996).

[17] Y. Xiao, Z. Y. Xu, L. Li, Z. H. Li, and G. S. Zhou, J. Nonlin. Optical Phys. Mater. **12**, 341 (2003).

[18] M. Nakazawa, H. Kubota, K. Suzuki, and E. Yamada, Chaos. **10**, 486 (2000).

[19] J. H. B. Nijhof, W. Forysiak, and N. J. Doran, IEEE J. Sel. Topics Quantum Electron. **6**, 330 (2000).

[20] A. Hasegawa, Y. Kodama, and A. Maruta, Opt. Tiber. Tech. **3**, 197 (1997).

[21] S. Kumar and A. Hasegawa, Opt. Lett. **22**, 372 (1997).

[22] K. J. Nakkeeran, J. Phys. A: Math. Gen. **34**, 5111 (2001).

[23] V. N. Serkin and A. Hasegawa, JETP Lett. **72**, 89 (2000).

[24] V. N. Serkin and A. Hasegawa, Phys. Rev. Lett. **85**, 4502 (2000).

[25] V. N. Serkin and A. Hasegawa, IEEE J. Sel. Topics Quantum Electron. **8**, 418 (2002).

[26] V. N. Serkin and T. L. Belyaeva, JETP Lett. **74**, 573 (2001).

[27] V. I. Kruglov, A. C. Peacock, and J. D. Harvey, Phys. Rev. Lett. **90**, 113902 (2003).

[28] S. A. Ponomarenko and G. P. Agrawal, Phys. Rev. Lett. **97**, 013901 (2006).